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BLOW-UP BEHAVIOR OF SOLUTIONS FOR SOME ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

Sarah Y. Bahk

September 2008

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September 2008

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Abstract

There are two parts in this project.

In part I, we consider the Riccati initial-value problem:

$$y'(t) = ay^2 + by + c,$$
$$y(0) = d,$$

where a, b, c, and d are real numbers and $t \geq 0$ represents time. We determine conditions on the constants a, b, c and d that are necessary and sufficient for $y(t)$ to approach either $+\infty$ or $-\infty$ as t approaches some finite value t_b.

In part II, we consider blow-up property of solutions for the degenerate semilinear parabolic initial-boundary value problem:

$$\xi^q u_\tau - u_{\xi\xi} = f(u(\xi_0, \tau)) \text{ for } 0 < \xi < a, 0 < \tau < \sigma,$$
$$u(\xi, 0) = u_0(\xi) \geq 0 \text{ for } 0 \leq \xi \leq a,$$
$$u(0, \tau) = 0 = u_\xi(a, \tau) \text{ for } \tau > 0.$$

Here a, σ, and q are constants with $a > 0, 0 < \sigma \leq \infty$, and $q \geq 0$. Also, let ξ_0 be some fixed point in $(0, a)$. It is assumed that $f \in C^2([0, \infty)), f(0) \geq 0, f' > 0, f'' \geq 0$. We will show that for sufficiently large initial function $u_0(\xi)$ solution of the above initial-boundary value problem blows up in finite time and the blow-up set is the entire interval $[0, a]$.

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I am grateful to my family for all their love and support in reaching my goals. I offer my deepest appreciation and gratitude to my husband, Chandler who has supported and encouraged me in many ways. Many thanks to my children, Gina, Rachel, and Joshua, for their patience and understanding during my work on this project. Finally special thanks to my parents, who sacrificed all their life in order to provide their children with advantages they never had.

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Chapter 1

Introduction

There are two parts in this project.

In part I, we consider the blow-up property of solutions for Riccati equation. Count Jacopolo Francesco Riccati (May 28, 1676 - April 15, 1754) is famous for introducing and researching solvability of the equation that is now known as Riccati equation:

$$y'(t) = a(t)y^2 + b(t)y + c(t). \tag{1.1}$$

The matrix form of this equation is very important in modern times since it is used extensively in design problems in filtering and control [Bit91]. Even though the Riccati equation (1.1) is not solvable in general, numerous methods are developed for finding solutions for special cases of this equation [PZ03].

We consider the Riccati initial-value problem:

$$\begin{cases} y'(t) = ay^2 + by + c, \\ y(0) = d, \end{cases} \tag{1.2}$$

where a, b, c, and d are real numbers and $t \geq 0$ represents time. We determine conditions on the constants a, b, c,and d that are necessary and sufficient for $y(t)$ to approach either $+\infty$ or $-\infty$ as t approaches some finite value t_b. We provide exact values for the time t_b for the cases when $4ac - b^2$ is positive, negative, or zero. We are interested in the first occurrence of blow-up. We do not consider behavior of $y(t)$ for $t > t_b$.

In part II, we consider blow-up property of solutions for the degenerate semilin-

ear parabolic initial-boundary value problem:

$$\begin{cases} \xi^q u_\tau - u_{\xi\xi} = f(u(\xi_0,\tau)) \quad \text{for} \quad 0<\xi<a, \quad 0<\tau<\sigma, \\ u(\xi,0) = u_0(\xi) \geq 0 \quad \text{for} \quad 0 \leq \xi \leq a, \\ u(0,\tau) = 0 = u_\xi(a,\tau) \quad \text{for} \quad \tau > 0. \end{cases}$$

Here a, σ, and q are constants with $a>0, 0<\sigma\leq\infty$, and $q\geq 0$. u_τ means the first order partial derivative of $u(\xi,\tau)$ with respect to τ, and $u_{\xi\xi}$ means the second order partial derivative of $u(\xi,\tau)$ with respect to ξ. Also let ξ_0 be some fixed point in $(0,a)$. Because the reaction term $f(u(\xi_0,\tau))$ depends on the fixed value ξ_0 in $(0,a)$, we say that this problem has localized nonlinear reaction. Let $\xi{=}ax, \tau{=}a^{q+2}t, D{=}(0,1), \Omega{=}D\times(0,T), \bar{D}$ and $\bar{\Omega}$ be the closures of D and Ω respectively, $x_0=\xi_0/a$, and $Lu = x^q u_t - u_{xx}$. We can see that $x_0=\xi_0/a$ is a fixed point in $(0,1)$. Since $\xi=ax$, we have $u_x = du/dx = (du/d\xi)\cdot(d\xi/dx) = u_\xi \cdot a$, then $u_\xi = u_x/a$. Also, $u_{xx} = d^2u/dx^2 = (d/dx)\cdot(du/dx) = (d/dx)(au_\xi)$, then we have $a\cdot(d/dx)(u_\xi) = a\cdot(du_\xi/d\xi)\cdot(d\xi/dx) = a\cdot u_{\xi\xi}\cdot a = a^2\cdot u_{\xi\xi}$, then we have $u_{\xi\xi} = u_{xx}/a^2$. Now since $\tau = a^{q+2}t$, so $u_t = du/dt = (du/d\tau)\cdot(d\tau/dt) = u_\tau \cdot a^{q+2}$, then we have $u_\tau = u_t/a^{q+2}$. And since $x_0=\xi_0/a$, then $\xi_0 = a\cdot x_0$. Now we have this:

$$\begin{aligned} \xi^q u_\tau - u_{\xi\xi} &= (ax)^q \frac{u_t}{a^{q+2}} - \frac{u_{xx}}{a^2} \\ &= \frac{x^q u_t - u_{xx}}{a^2}. \end{aligned}$$

Since $\xi = ax$, and $0<\xi<a$, then $0<ax<a$, so we have $0<x<1$. Also, since $\tau = a^{q+2}t$, and $0<\tau<\sigma$, then $0<a^{q+2}t<\sigma$, so $0<t<\sigma/\left(a^{q+2}\right)=T$.
So $f\left(u(\xi_0,\tau)\right) = f(u(ax_0,a^{q+t}t)) = [x^q\cdot u_t(ax_0,a^{q+2}t) - u_{xx}(ax_0,a^{q+2}t)]/a^2 = f\left(u(x_0,t)\right)$ in $\Omega, (0<x<1, 0<t<T)$. Then, $\left(x^q u_t - u_{xx}\right)/a^2 = f\left(u(x_0,t)\right)$, hence $x^q u_t - u_{xx} = a^2 f(u(x_0,t))$.

The above problem is transformed into

$$\begin{cases} Lu = a^2 f(u(x_0,t)) & \text{in} \quad \Omega, \\ u(x,0) = u_0(x) \geq 0 & \text{on} \quad \bar{D}, \\ u(0,t) = 0 = u_x(1,t) & \text{for} \quad 0<t<T, \end{cases} \tag{1.3}$$

where $T{=}\sigma/a^{q+2} \leq \infty$. We assume that $f\in C^2([0,\infty)), f(0)\geq 0, f'>0, f''\geq 0$, and there exists a positive number r such that

$$f(u) \geq u^{1+r} \quad \text{for} \quad u \geq 1. \tag{1.4}$$

The function $u_0(x) \in C^{2+\alpha}(\bar{D})$ is required to satisfy the compatibility conditions $u_0(0)=0$ and $u_0'(1) = 0$, where $0 < \alpha < 1$. The solution of the problem (1.3) is said to blow-up at $x = \bar{x}$ and $t = t_b$ if there exists a sequence $\{(x_n, t_n)\} \to (\bar{x}, t_b)$ and $lim_{n\to\infty} u(x_n, t_n) \to \infty$. The blow-up of u is complete at t_b if at t_b, u blows up at every point $x \in \bar{D}$.

The complete blow-up of the solution of the problem (1.3) with $u_x(1, t)$ replaced by $u(1, t)$ was investigated by Chan and Yang [CY00]. In Chapter 3, we show existence of a unique classical solution of the problem (1.3) for any $q \geq 0$. If $T < \infty$, then $u(x_0, t)$ is unbounded in $(0, T)$. A criterion for u to blow up in a finite time is also given, and a nonlinear integral equation in terms of Green's function is used to show that the localized nonlinear reaction leads to the complete blow-up of u for any $q \geq 0$.

Chapter 2

Riccati Problems with Constant Coefficients

We will start by stating the theorem which is very important for the Riccati problems.

Theorem 2.1. *The following is true for the solution $y(t)$ of (1.2):*
1. Let $4ac - b^2 > 0$. If $a > 0$, then $y(t) \to +\infty$, while if $a < 0$, then $y(t) \to -\infty$.
2. Let $4ac - b^2 = 0$. If $a > 0$, and $d > -\frac{b}{2a}$, then $y(t) \to +\infty$. If $a < 0$, and $d < -\frac{b}{2a}$, then $y(t) \to -\infty$. Otherwise, $y(t)$ is bounded for any finite $t > 0$. In particular, if $d = -\frac{b}{2a}$, then $y(t) \equiv d$.
3. Let $4ac - b^2 < 0$. If $a > 0$ and $d > \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, then $y(t) \to +\infty$. If $a < 0$ and $d < \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, then $y(t) \to -\infty$. Otherwise, $y(t)$ is bounded for any finite $t > 0$. In particular, if $d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, then $y(t) \equiv d$.
4. If $a = 0$, then $y(t)$ is bounded for all $t > 0$.

Proof. 1. Let $4ac - b^2 > 0$. A solution of the initial value problem (1.2) can be found using separation of variables and the table of integrals [LHE06]:

$$y(t) = \frac{\sqrt{4ac - b^2}}{2a} \tan\left[\frac{t\sqrt{4ac - b^2}}{2} + \arctan\left(\frac{b + 2ad}{\sqrt{4ac - b^2}}\right)\right] - \frac{b}{2a}. \tag{2.1}$$

We can find the blow-up time t_b by solving the following equation for t:

$$\frac{t\sqrt{4ac-b^2}}{2} + \arctan\left(\frac{b+2ad}{\sqrt{4ac-b^2}}\right) = \frac{\pi}{2},$$

$$t_b = \frac{\pi}{\sqrt{4ac-b^2}} - \frac{2}{\sqrt{4ac-b^2}}\arctan\left[\frac{b+2ad}{\sqrt{4ac-b^2}}\right].$$

Also,

$$-\frac{\pi}{2} < \arctan\left[\frac{b+2ad}{\sqrt{4ac-b^2}}\right] < \frac{\pi}{2}.$$

This implies that t_b is always positive and solution $y(t)$ of (1.2) is guaranteed to blow-up as t approaches t_b. Also, from equation (2.1) we notice that if $a > 0$, then $y(t) \to +\infty$, while if $a < 0$, then $y(t) \to -\infty$. Changing initial value d cannot prevent blow-up from occurring. However, d influences the blow-up time t_b. For example, if $a < 0$, then decreasing d will accelerate the blow-up. If $a > 0$, then increasing d will accelerate the blow-up.

2. Let $4ac - b^2 = 0$. Using separation of variables, we obtain:

$$\int \frac{dy}{a\left(y + \frac{b}{2a}\right)^2} = \int dt.$$

Integration leads to the following solution:

$$y(t) = \frac{2ad+b}{a(2-2adt-bt)} - \frac{b}{2a}. \tag{2.2}$$

To find the blow-up time we will set the denominator of the first term in (2.2) equal to 0:

$$2 - 2adt - bt = 0,$$

$$t_b = \frac{2}{2ad+b}.$$

From the inequality

$$t_b = \frac{2}{2ad+b} > 0,$$

and from (2.2) we obtain the following: if $a > 0$ and $d > -\frac{b}{2a}$, then $y(t) \to +\infty$, while if $a < 0$ and $d < -\frac{b}{2a}$, then $y(t) \to -\infty$, Initial value d is very important since

certain values can prevent blow-up from occurring. Also, d influences the blow-up time t_b. If blow-up occurs for some value d, then decreasing d (if $a < 0$) or increasing d (if $a > 0$) will accelerate the blow-up. Also, if $d = -\frac{b}{2a}$, then $y(t) \equiv d$ satisfies initial-value problem (1.2). By the existence and uniqueness theorem for first order initial-value problem [BDH02], (1.2) has a unique solution. Therefore, in this special case $y(t)$ is bounded for all finite $t > 0$.

3. Let $4ac - b^2 < 0$. Let us notice that if

$$d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

then $y \equiv d$ is the solution of the initial-value problem (1.2). Therefore, in this case solution is bounded for all $t > 0$. Now let us consider the case when

$$d \neq \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Using separation of variables and the table of integrals [LHE06], we have:

$$\int \frac{dy}{ay^2 + by + c} = \int dt,$$

$$\frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ay + b - \sqrt{b^2 - 4ac}}{2ay + b + \sqrt{b^2 - 4ac}} \right| = t + C_1, \tag{2.3}$$

where C_1 is a constant of integration. We can find C_1 substituting the initial condition $y(0) = d$ into the equation (2.3). We have:

$$C_1 = \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ad + b - \sqrt{b^2 - 4ac}}{2ad + b + \sqrt{b^2 - 4ac}} \right|. \tag{2.4}$$

We will consider two possible cases:

$$\frac{2ad + b - \sqrt{b^2 - 4ac}}{2ad + b + \sqrt{b^2 - 4ac}} > 0 \tag{2.5}$$

and

$$\frac{2ad + b - \sqrt{b^2 - 4ac}}{2ad + b + \sqrt{b^2 - 4ac}} < 0. \tag{2.6}$$

Let us substitute (2.4) into (2.3) and solve for $y(t)$. In case (2.5), we can omit the absolute value symbol:

$$\frac{1}{\sqrt{b^2-4ac}}\ln\left(\frac{2ay+b-\sqrt{b^2-4ac}}{2ay+b+\sqrt{b^2-4ac}}\right)= t+\frac{1}{\sqrt{b^2-4ac}}\ln\left(\frac{2ad+b-\sqrt{b^2-4ac}}{2ad+b+\sqrt{b^2-4ac}}\right).$$

In case (2.6), we have:

$$\frac{1}{\sqrt{b^2-4ac}}\ln\left(-\frac{2ay+b-\sqrt{b^2-4ac}}{2ay+b+\sqrt{b^2-4ac}}\right)= t+\frac{1}{\sqrt{b^2-4ac}}\ln\left(-\frac{2ad+b-\sqrt{b^2-4ac}}{2ad+b+\sqrt{b^2-4ac}}\right).$$

In both cases (2.5) and (2.6), the solution of the problem (1.2) is given by the following formula:

$$y(t)=\frac{-bd+d\sqrt{b^2-4ac}-2c+(bd+d\sqrt{b^2-4ac}+2c)e^{t\sqrt{b^2-4ac}}}{2ad+b+\sqrt{b^2-4ac}-(2ad+b-\sqrt{b^2-4ac})e^{t\sqrt{b^2-4ac}}}. \tag{2.7}$$

To find the blow-up time we have to set the denominator equal to 0:

$$2ad+b+\sqrt{b^2-4ac}-(2ad+b-\sqrt{b^2-4ac})e^{t\sqrt{b^2-4ac}}=0. \tag{2.8}$$

Solving equation (2.8) for t, we obtain:

$$t_b=\frac{1}{\sqrt{b^2-4ac}}\ln\left(\frac{2ad+b+\sqrt{b^2-4ac}}{2ad+b-\sqrt{b^2-4ac}}\right).$$

The blow-up time t_b must be positive, therefore, we have:

$$\frac{2ad+b+\sqrt{b^2-4ac}}{2ad+b-\sqrt{b^2-4ac}}>1. \tag{2.9}$$

Let us observe that if equation (2.6) holds, then (2.8) can never be satisfied, therefore, there is no blow-up. If equation (2.5) holds, then there are two possibilities:

$$\begin{cases} 2ad+b-\sqrt{b^2-4ac}>0 \\ 2ad+b+\sqrt{b^2-4ac}>0 \end{cases} \tag{2.10}$$

or

$$\begin{cases} 2ad + b - \sqrt{b^2 - 4ac} < 0 \\ 2ad + b + \sqrt{b^2 - 4ac} < 0. \end{cases} \tag{2.11}$$

Solving (2.10) and (2.9) simultaneously, we obtain conditions on d that lead to blow-up in finite time:

$$d > \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ if } a > 0,$$
$$d < \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ if } a < 0.$$

Solving (2.11) and (2.9) simultaneously, we obtain a contradiction which implies that there is no blow-up in this case. We notice that from (2.7) that if $a > 0$ and $d > \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, then $y(t) \to +\infty$. if $a < 0$ and $d < \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, then $y(t) \to -\infty$. Initial value d is very important since certain values can prevent blow-up from occurring. Also, d influences the blow-up time t_b.

4. If $a = 0$, then the equation is linear. Using separation of variables, we obtain:

$$\int \frac{dy}{by + c} = \int dt,$$

$$y(t) = \frac{(bd + c)e^{bt} - c}{b}.$$

Function $y(t)$ is bounded for any finite time $t > 0$.

□

Example 1 (case 3): Let us investigate the blow-up property of the solution for the initial-value problem

$$\begin{cases} y'(t) = -4y^2 + 5y - 1 \\ y(0) = d, \end{cases} \tag{2.12}$$

with three different values of d as indicated below.

First, we notice that $a = -4 < 0, b = 5, c = -1$, and $4ac - b^2 = -9 < 0$. Also, $(-b + \sqrt{b^2 - 4ac})/(2a) = 0.25$. According to the Theorem 1, we expect that solution of the problem (2.12) blows up for any $d < 0.25$ and is bounded otherwise. Solution of the problem (2.12) is given by the formula (2.7).

If $d = 2$, then we have

$$y(t) = \frac{-2 + 14e^{3t}}{-8 + 14e^{3t}}.$$

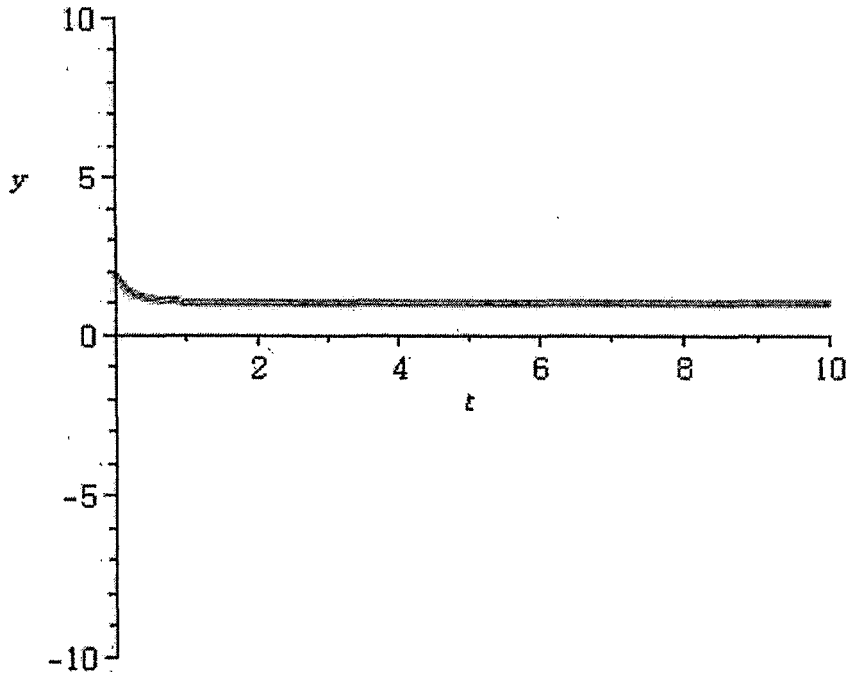

Figure 2.1: example of the case $4ac - b^2 < 0, \quad a < 0, \quad d > (-b + \sqrt{b^2 - 4ac})/(2a)$

This function is bounded for any finite time $t > 0$.

If $d = 0$, then we have

$$y(t) = \frac{1 - e^{3t}}{4 - e^{3t}}.$$

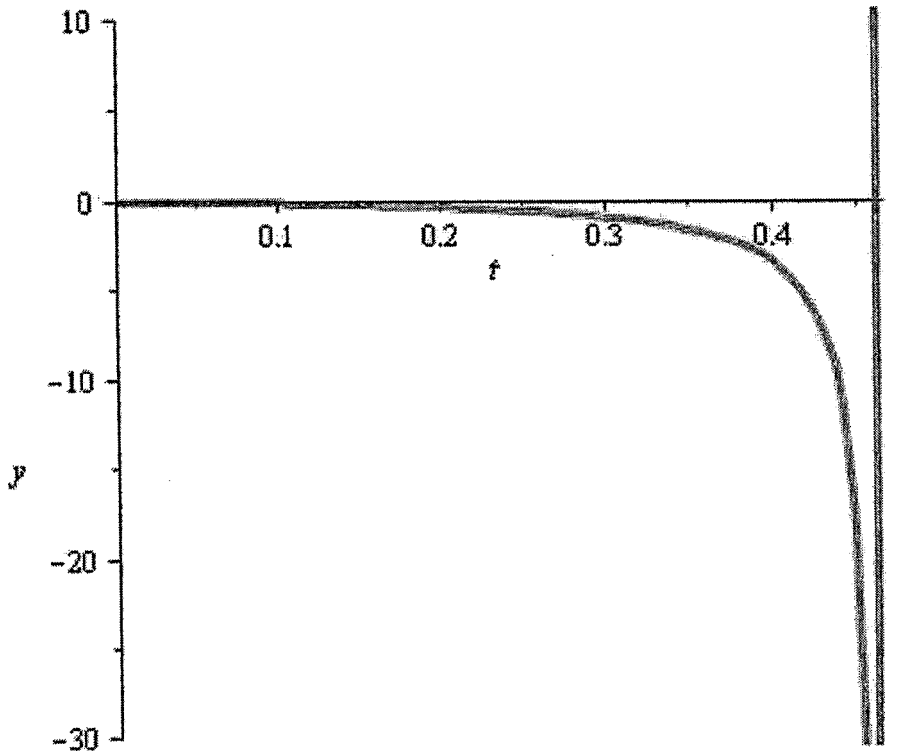

Figure 2.2: example of the case $4ac - b^2 < 0, \quad a < 0, \quad d < (-b + \sqrt{b^2 - 4ac})/(2a)$

Function $y(t) \to -\infty$ when $t_b = \ln(4)/3$.

Example 2 (case 3):

$$\begin{cases} y'(t) = 4y^2 + 5y + 1, \\ y(0) = d. \end{cases} \tag{2.13}$$

First, we notice that $a = 4 > 0, b = 5, c = 1, d = 0$, and $4ac - b^2 = -9 < 0$. Also, $(-b + \sqrt{b^2 - 4ac})/(2a) = -0.25$. According to the Theorem 1, we expect that solution of the problem (2.13) blows up for any $d > -0.25$ and is bounded otherwise. Solution of the problem (2.13) is given by the formula (2.7). If $d = 0$, then we have

$$y(t) = \frac{-2 + 2e^{3t}}{8 - 2e^{3t}}.$$

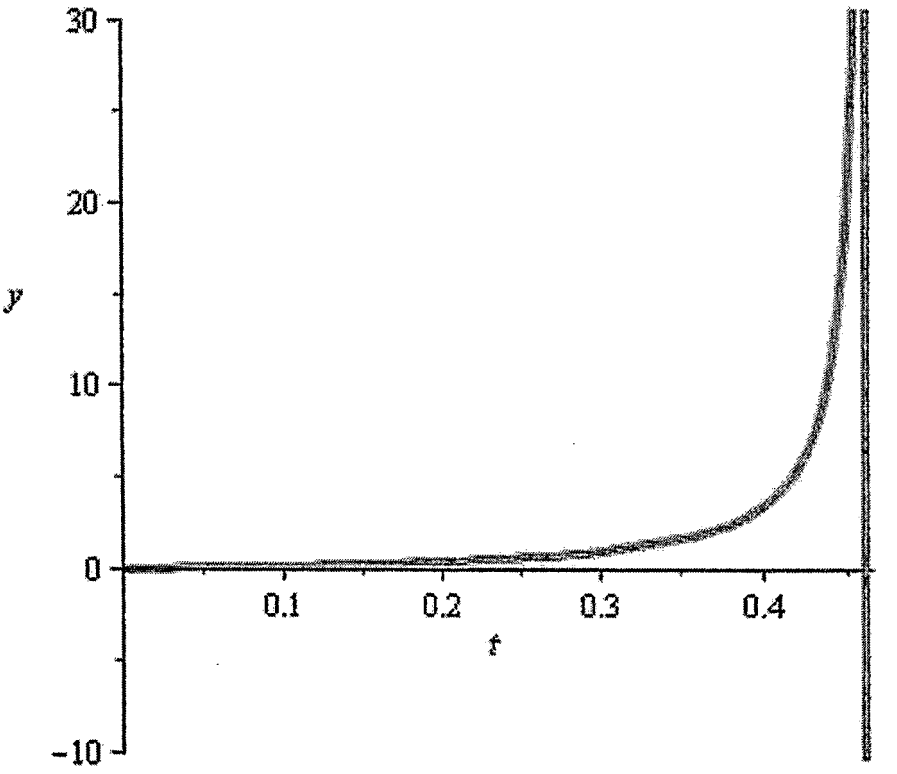

Figure 2.3: example of the case $4ac - b^2 < 0, \quad a > 0, \quad d > (-b + \sqrt{b^2 - 4ac})/(2a)$

Function $y(t) \to +\infty$ when $t_b = \ln(4)/3$.

Example 3 (case 2):

$$\begin{cases} y'(t) = y^2 + 2y + 1, \\ y(0) = d. \end{cases} \tag{2.14}$$

First, we notice that $a = 1 > 0, b = 2, c = 1$, and $4ac - b^2 = 0$. Also, $-b/2a = -1$. According to the Theorem 1, we expect that solution of the problem (2.14) approaches $+\infty$ for any $d > -1$ and is bounded otherwise. Solution of the problem (2.14) is given by the formula (2.2). If $d = 0$, then we have

$$y(t) = \frac{1}{1-t} - 1.$$

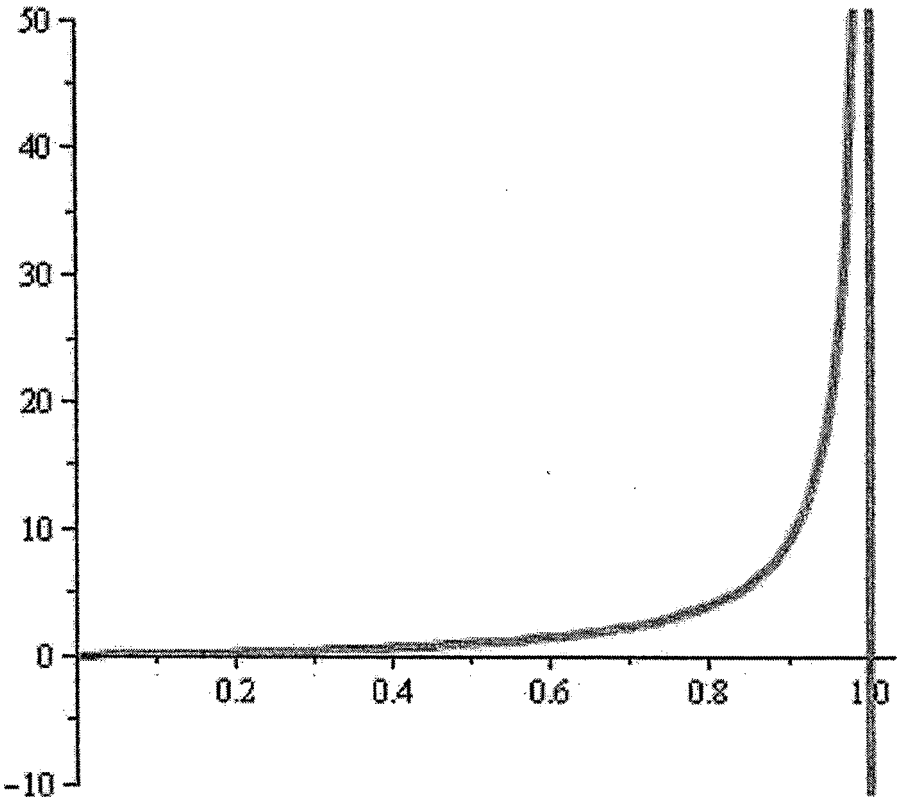

Figure 2.4: example of the case $4ac - b^2 = 0, \quad a > 0, \quad d > -b/2a$

Function $y(t) \to +\infty$ when $t_b = 1$.

$$\begin{cases} y'(t) = -y^2 - 2y - 1, \\ y(0) = d. \end{cases} \tag{2.15}$$

First, we notice that $a = -1 < 0, \quad b = -2, \quad c = -1$, and $4ac - b^2 = 0$. Also, $-b/2a = -1$. According to the Theorem 1, we expect that solution of the problem (2.15) approaches $-\infty$ for any $d < -1$ and is bounded otherwise. Solution of the problem (2.15) is given by the formula (2.2).

If $d = -1.5$, then we have

$$y(t) = \frac{1}{-1+t} - 1.$$

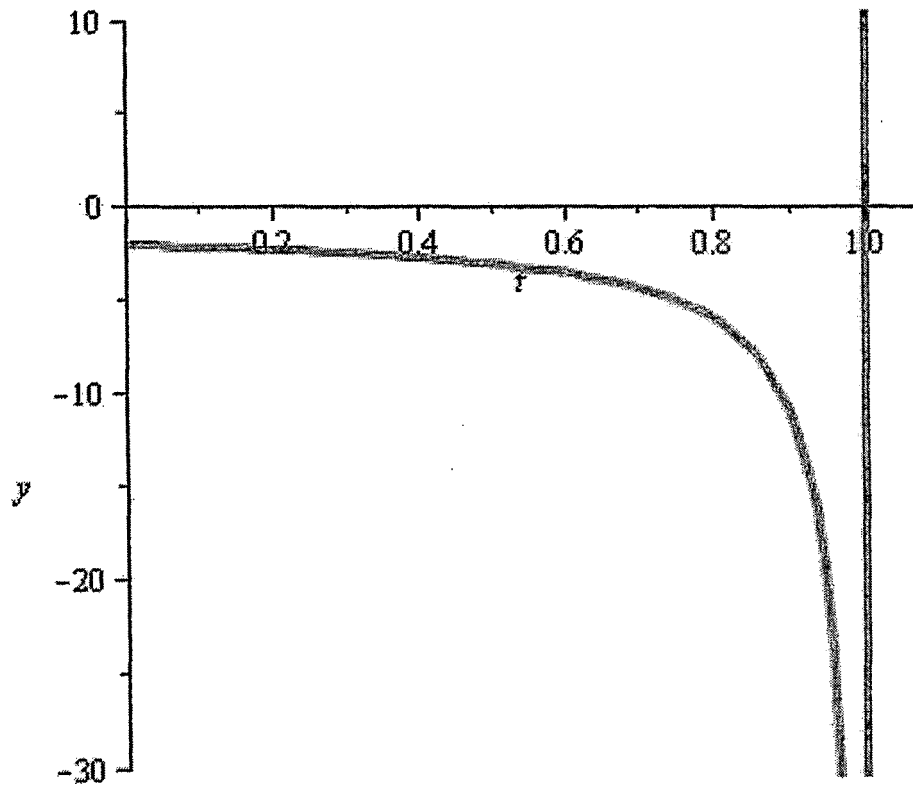

Figure 2.5: example of the case $4ac - b^2 = 0, \quad a < 0, \quad d < -b/2a$

Function $y(t) \to -\infty$ when $t_b = 1$.

Example 4 (case 1):

$$\begin{cases} y'(t) = 2y^2 + 3y + 2, \\ y(0) = d. \end{cases}$$

First we notice that $a = 2 > 0, b = 3, c = 2, d = 1$ and $4ac - b^2 = 7 > 0$. According to the Theorem 1, we expect $y(t) \to +\infty$ since $a > 0$. Changing initial value d cannot prevent blow-up from occurring.

$$y(t) = \frac{\sqrt{7}}{4}\tan\left[\frac{t\sqrt{7}}{2} + \arctan(\sqrt{7})\right] - \frac{3}{4}.$$

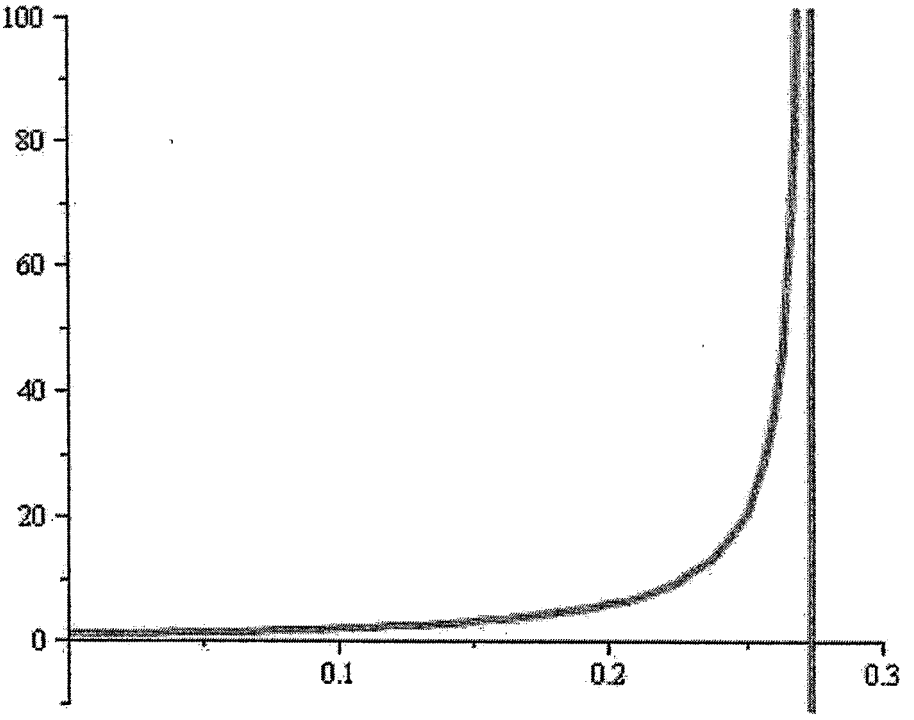

Figure 2.6: example of the case $4ac - b^2 > 0, \quad a > 0$

Function $y(t) \to +\infty$ when $t > 0$.

$$\begin{cases} y'(t) = -3y^2 + 2y - 2, \\ y(0) = d. \end{cases}$$

First we notice that $a = -3 < 0, b = 2, c = -2, d = 1$ and $4ac - b^2 = 20 > 0$. According to the Theorem 1, we expect $y(t) \to -\infty$ since $a < 0$. Changing initial value d cannot prevent blow-up from occurring.

$$y(t) = \frac{\sqrt{5}}{-3}\tan\left[t\sqrt{5} + \arctan\left(\frac{-2}{\sqrt{5}}\right)\right] + \frac{1}{3}.$$

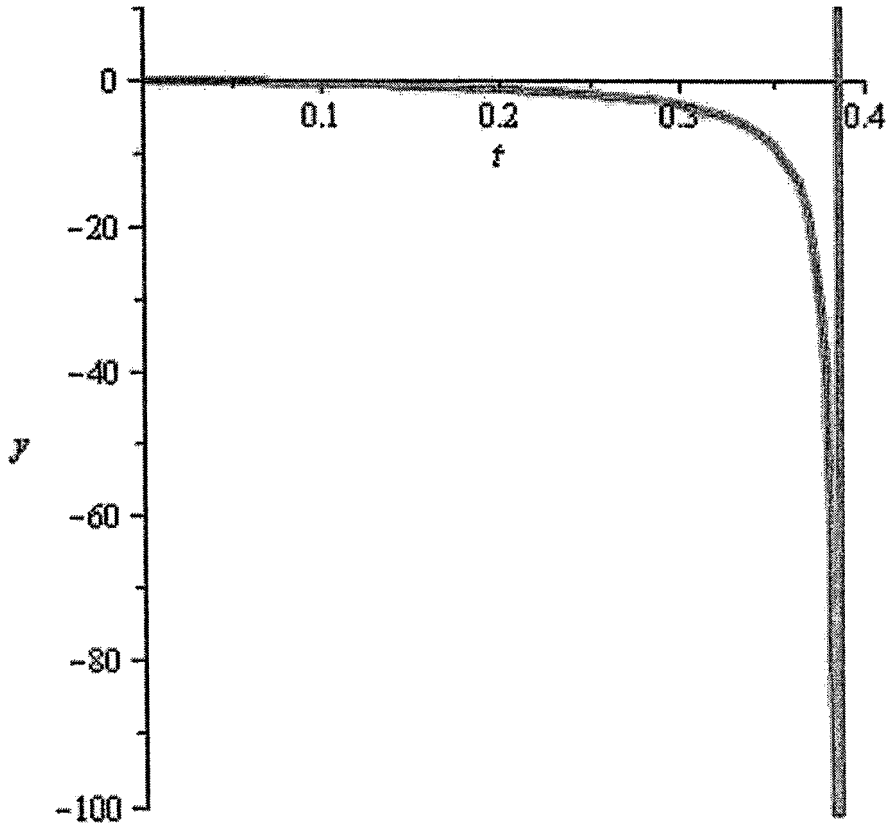

Figure 2.7: example of the case $4ac - b^2 > 0, \quad a < 0$

Function $y(t) \to -\infty$ when $t > 0$.

Proof. If $b(x,t) \equiv 0$, we can apply the strong maximum principle [Fri64] to obtain the conclusion immediately.

For the case $b(x,t)$ being nonnegative and nontrivial (not indentically zero), let η be a positive constant, and

$$V(x,t) = u(x,t) + \eta(1 + x^{1/2})e^{ct},$$

where c is a positive constant to be determined.

Let us verify $V(x,0) > 0$ on $\partial\Omega$. We have $V(x,0) = u(x,0) + \eta(1 + x^{1/2})e^0 > 0$, because we know that $u(x,0) \geq 0$, and $\eta(1 + x^{1/2})e^0 > 0$, thus $V(x,0) > 0$.

Since $u(0,t) = 0$ and $\eta(1+0)e^{ct} > 0$, so $V(0,t) = u(0,t) + \eta(1+0)e^{ct} > 0$. As we know $\xi = ax$, then we have $u_\xi = u_x/a$. We know $u_\xi(a,\tau) = 0$, for $\tau > 0$. By the substitution, we get $(u_x/a)(1,t) = 0$, for $0 < t < T$, finally we have $u_x(1,t) = 0$.

We have

$$\begin{aligned} V_x(x,t) &= u_x(x,t) + \frac{1}{2}\eta e^{ct} x^{1/2-1}, \\ &= u_x(x,t) + \frac{\eta e^{ct}}{2\sqrt{x}}, \end{aligned}$$

thus $V_x(1,t) = u_x(1,t) + \dfrac{\eta e^{ct}}{2} > 0$.

Let $\partial\Omega$ denote the parabolic boundary

$$(\{0,a\} \times (0,T)) \cup ([0,a] \times \{0\})$$

of Ω. Then $V(x,t) > 0$ on $\partial\Omega$, and $Lu - b(x,t)u(x_0,t) \geq 0$.

Now,

$$\begin{aligned} &LV - b(x,t)V(x_0,t) = L(u(x,t) + \eta(1+\sqrt{x})e^{ct}) - b(x,t)V(x_0,t) \\ &= x^q \frac{\partial}{\partial t}(u + \eta(1+\sqrt{x})e^{ct}) - \frac{\partial^2}{\partial x^2}(u + \eta(1+\sqrt{x})e^{ct}) - b(x,t)[u(x_0,t) + \eta(1 + x_0^{1/2})e^{ct}] \\ &= Lu - b(x,t)u(x_0,t) + L(\eta(1+\sqrt{x})e^{ct}) - b(x,t)\eta(1+\sqrt{x_0})e^{ct} \\ &\geq L(\eta(1+\sqrt{x})e^{ct}) - b(x,t)\eta(1+\sqrt{x_0})e^{ct} \\ &= \frac{x^q \partial(\eta(1+\sqrt{x})e^{ct})}{\partial t} - \frac{\partial^2(\eta(1+\sqrt{x})e^{ct})}{\partial x^2} - b(x,t)\eta(1+\sqrt{x_0})e^{ct} \\ &= x^q \eta(1+\sqrt{x})e^{ct}c - \frac{\partial}{\partial x}\left(\frac{\eta e^{ct}}{2\sqrt{x}}\right) - b(x,t)\eta(1+\sqrt{x_0})e^{ct} \\ &= \eta e^{ct}\left((cx^q(1+\sqrt{x})) - b(x,t)(1+\sqrt{x_0}) + \frac{1}{4x^{3/2}}\right). \end{aligned}$$

Let s denote the positive zero of

$$\frac{1}{4x^{3/2}} - (1+x_0^{1/2})\max_{(x,t)\in\Omega} b(x,t).$$

For $x = s$,

$$\begin{aligned}
&cx^q(1+x^{1/2}) - b(x,t)(1+x_0^{1/2}) + \frac{1}{4x^{3/2}} \\
&\geq cx^q(1+x^{1/2}) - (1+x_0^{1/2})\max_{(x,t)\in\Omega} b(x,t) + \frac{1}{4x^{3/2}} \\
&= cs^q(1+s^{1/2}) - (1+x_0^{1/2})\max_{(x,t)\in\Omega} b(x,t) + \frac{1}{4s^{3/2}} \\
&= cs^q(1+s^{1/2}) - 0 > 0.
\end{aligned}$$

For $x < s$, since $cx^q(1+x^{1/2}) \geq 0$, and $-(1+x^{1/2})\max_{(x,t)\in\Omega} b(x,t) + \frac{1}{4x^{3/2}} > 0$,

$$\begin{aligned}
&cx^q(1+x^{1/2}) - b(x,t)(1+x_0^{1/2}) + \frac{1}{4x^{3/2}} \\
&\geq cx^q(1+x^{1/2}) - (1+x_0^{1/2})\max_{(x,t)\in\Omega} b(x,t) + \frac{1}{4x^{3/2}} > 0.
\end{aligned}$$

For $x > s$,

$$\begin{aligned}
&cx^q(1+x^{1/2}) - b(x,t)(1+x_0^{1/2}) + \frac{1}{4x^{3/2}} \\
&\geq cx^q(1+x^{1/2}) - (1+x_0^{1/2})\max_{(x,t)\in\Omega} b(x,t) + \frac{1}{4x^{3/2}} \\
&> cs^q - (1+x_0^{1/2})\max_{(x,t)\in\Omega} b(x,t) \geq 0.
\end{aligned}$$

If we choose $c \geq \dfrac{(1+x_0^{1/2})\max_{(x,t)\in\Omega} b(x,t)}{s^q}$, then $LV - b(x,t)V(x_0,t) > 0$ in Ω.

Suppose $V(x,t) \leq 0$ somewhere in Ω. Then, the set

$$\{t : V(x,t) \leq 0 \text{ for some } x \in D\}$$

is nonempty. Let $\bar{t}$ denote its infimum. Since $V(x,0) > 0$, we have $0 < \bar{t} < T$. Thus, there exists some $x_1 \in D$ such that $V(x_1,\bar{t}) = 0$, and $V_t(x_1,\bar{t}) \leq 0$. On the other hand, since $V(x,t)$ attains its local minimum at $(x_1,\bar{t})$, we have $V_{xx}(x_1,\bar{t}) \geq 0$. Since $\bar{t}$ is the infimum, we also have $V(x_0,\bar{t}) \geq 0$. Now we have,

$$\begin{aligned}
LV(x_1,\bar{t}) = x_1^q V_t(x_1,\bar{t}) - V_{xx}(x_1,\bar{t}) \leq 0, \\
x_1^q V_t(x_1,\bar{t}) - V_{xx}(x_1,\bar{t}) - b(x_1,\bar{t})V(x_0,\bar{t}) \leq 0, \\
0 \geq x_1^q V_t(x_1,\bar{t}) \geq LV(x_1,\bar{t}) - b(x_1,\bar{t})V(x_0,\bar{t}) > 0.
\end{aligned}$$

This contradiction shows that $V(x,t) = u(x,t) + \eta(1+x^{1/2}) > 0$, in Ω.
As $\eta \to 0^+$, $u(x,t) \geq 0$.

□

3.2 Existence and Uniqueness

Let $\omega = D \times (0, t_0)$ for some positive number t_0, and $\bar{\omega}$ be its closure.

Lemma 3.2. *There exists a positive constant $t_0 < T$ such that the problem (1.3) has an upper solution $\mu_1(x,t) \in C^{2,1}(\bar{\omega})$.*

Note: An upper solution $\mu_1(x,t)$ has to satisfy the following:

$$\begin{aligned} &L\mu_1 - a^2 f(\mu_1(x_0,t)) \geq 0 \quad \text{in} \quad \omega \\ &\mu_1(x,0) \geq u_0(x) \quad \text{on} \quad \bar{D} \\ &\mu_{1_x}(1,t) \geq 0, \quad t \in [0,t_0] \\ &\mu_1(0,t) \geq 0, \quad t \in [0,t_0]. \end{aligned}$$

Note: $\mu_1 \in C^{2,1}(\bar{\omega})$ means that $\mu_1, \mu_{1_x}, \mu_{1_{xx}}$ and μ_{1_t} are continuous on $\bar{\omega}$.

Proof. Let $k_1 = 1 + \max_{x\in\bar{D}}(u_0(x))$, and $k_2 (> 2a^2)$ be chosen sufficiently large such that

$$\gamma \equiv k_2 f(1+k_1) > 2, \tag{3.2}$$

$$0 < \epsilon < \min\left\{1 - \frac{1}{2^{\frac{1}{\gamma-2}}}, \frac{1}{k_2}\sqrt{\frac{k_2 - 2a^2}{f(1+k_1)}}\right\}. \tag{3.3}$$

Since $2^{1/(\gamma-2)} > 1$ for any $\gamma > 2$, we have

$$0 < 1 - \frac{1}{2^{\frac{1}{\gamma-2}}} < 1.$$

Let $\mu_1(x,t) = \theta_1(x)\tau_1(t)$, where

$$\theta_1(x) = (1-x)^\gamma e^{\gamma x} + k_1, \tag{3.4}$$

$$\tau_1' = \epsilon^{-q} k_1^{-1} \left[\frac{\gamma(1+\gamma)e^{\gamma}}{\theta_1(x_0)} + a^2 \right] f(\theta_1(x_0)\tau_1), \quad \tau_1(0) = 1. \tag{3.5}$$

By the existence and uniqueness theorem for first order initial-value problem [BDH02], (3.5) has a unique solution. We note that in D,

$$\begin{aligned} \theta_1'(x) &= \gamma(1-x)^{\gamma-1}(-1)e^{\gamma x} + (1-x)^{\gamma}\gamma e^{\gamma x} \\ &= -\gamma(1-x)^{\gamma-1} x e^{\gamma x} < 0, \end{aligned}$$

and the function $\theta_1(x)$ is decreasing, which implies that

$$\theta_1(x) \leq 1 + k_1.$$

From (3.2),

$$\begin{aligned} &\gamma - \epsilon^2\gamma^2\tau_1(0) - \frac{a^2 f(\theta_1(x_0)\tau_1(0))}{(1-\epsilon)^{\gamma-2}} \\ &= \gamma - \epsilon^2\gamma^2 - \frac{a^2 f(\theta_1(x_0))}{(1-\epsilon)^{\gamma-2}} \\ &\geq k_2 f(1+k_1) - \epsilon^2 k_2^2 f^2(1+k_1) - \frac{a^2 f(1+k_1)}{\left(1 - 1 + \frac{1}{2^{\frac{1}{\gamma-2}}}\right)^{\gamma-2}} \\ &= f(1+k_1)(k_2 - 2a^2 - \epsilon^2 k_2^2 f(1+k_1)) \\ &> 0. \end{aligned}$$

We used the following: $\left(\frac{1}{2^{\frac{1}{\gamma-2}}}\right)^{\gamma-2} = \frac{1^{\gamma-2}}{\left(2^{\frac{1}{\gamma-2}}\right)^{\gamma-2}} = \frac{1}{2^{\frac{\gamma-2}{\gamma-2}}} = \frac{1}{2}$, and

$$\begin{aligned} &k_2 - 2a^2 - \epsilon^2 k_2^2 f(1+k_1) > 0, \\ &k_2 - 2a^2 > \epsilon^2 k_2^2 f(1+k_1), \\ &\frac{k_2 - 2a^2}{k_2^2 f(1+k_1)} > \epsilon^2, \\ &\epsilon < \sqrt{\frac{k_2 - 2a^2}{k_2^2 f(1+k_1)}} = \frac{1}{k_2}\sqrt{\frac{k_2 - 2a^2}{f(1+k_1)}} \end{aligned}$$

holds by (3.3).

Also, $\tau_1(t)$ is an increasing function that blows up at

$$t_b = \frac{\epsilon^q k_1}{\gamma(1+\gamma)e^{\gamma} + a^2\theta_1(x_0)} \int_{\theta_1(x_0)}^{\infty} \frac{ds}{f(s)} < \infty.$$

Let t_2 denote the time such that

$$\gamma - \epsilon^2\gamma^2\tau_1(t_2) - \frac{a^2 f(\theta_1(x_0)\tau(t_2))}{(1-\epsilon)^{\gamma-2}} = 0, \tag{3.6}$$

and $t_0 = \min\{t_2; t_b - \kappa\}$ for some fixed small number $0 < \kappa \ll t_b$. Since $\tau_1(t)$ is an increasing function, it follows from (3.2), (3.4), and (3.6) that for any $x \in [0, \epsilon]$ and $t \leq t_0$,

$$\begin{aligned}
&L\mu_1 - a^2 f(\mu_1(x_0,t)) = x^q \mu_{1_t} - \mu_{1_{xx}} - a^2 f(\mu_1(x_0,t)) \\
&= x^q \theta_1(x)\tau_1'(t) - \theta_1''(x)\tau_1(t) - a^2 f(\theta_1(x_0)\tau_1) \\
&= \Big(x^q\big((1-x)^\gamma e^{\gamma x} + k_1\big)\Big)\left(\epsilon^{-q}k_1^{-1}\left[\frac{\gamma(1+\gamma)e^\gamma}{\theta_1(x_0)} + a^2\right] f(\theta_1(x_0)\tau_1)\right) \\
&-\Big(\gamma(1-x)^{\gamma-2}(-1+\gamma x^2)e^{\gamma x}\Big)\tau_1 - a^2 f(\theta_1(x_0)\tau_1) \\
&\geq -\gamma(1-x)^{\gamma-2}(-1+\gamma x^2)e^{\gamma x}\tau_1 - a^2 f(\theta_1(x_0)\tau_1) \\
&\geq (1-x)^{\gamma-2}e^{\gamma x}\left[\gamma\tau_1(0) - \gamma^2\tau_1(t_0)x^2 - \frac{a^2 f(\theta_1(x_0)\tau_1(t_0))}{e^{\gamma x}(1-x)^{\gamma-2}}\right] \\
&\geq (1-x)^{\gamma-2}e^{\gamma x}\left[\gamma - \gamma^2\tau_1(t_0)\epsilon^2 - \frac{a^2 f(\theta_1(x_0)\tau_1(t_0))}{(1-\epsilon)^{\gamma-2}}\right] \geq 0.
\end{aligned}$$

From (3.4) we get

$$\begin{aligned}
\theta_1''(x) &= -\gamma(\gamma-1)(1-x)^{\gamma-2}(-1)xe^{\gamma x} - \gamma(1-x)^{\gamma-1}e^{\gamma x} - \gamma(1-x)^{\gamma-1}x\gamma e^{\gamma x} \\
&= \gamma(1-x)^{\gamma-2}(-1+x^2\gamma)e^{\gamma x},
\end{aligned}$$

we obtain

$$\max_{x\in\bar{D}} \left|\theta_1''(x)\right| \leq \gamma(1+\gamma)e^\gamma.$$

Using $\tau_1' > 0$, and from (1.4), (3.2), and (3.6) we have $f(\theta_1(x_0)\tau_1) \geq \theta_1(x_0)\tau_1$,

and for any $x \in [\epsilon, 1]$, and $t \leq t_0$,

$$
\begin{aligned}
&L\mu_1 - a^2 f(\mu_1(x_0,t)) = x^q \mu_{1_t} - \mu_{1_{xx}} - a^2 f(\mu_1(x_0,t)) \\
&= x^q \theta_1(x)\tau_1'(t) - \theta_1''(x)\tau_1(t) - a^2 f(\mu_1(x_0,t)) \\
&= x^q \Big((1-x)^\gamma e^{\gamma x} + k_1\Big)\tau_1'(t) - \Big(\gamma(1-x)^{\gamma-2}(-1+\gamma x^2)e^{\gamma x}\Big)\tau_1(t) - a^2 f(\mu_1(x_0,t)) \\
&\geq x^q \Big((1-x)^\gamma e^{\gamma x} + k_1\Big)\tau_1'(t) - \Big(\gamma(1+\gamma)e^{\gamma}\Big)\tau_1(t) - a^2 f(\mu_1(x_0,t)) \\
&\geq \epsilon^q k_1 \tau_1' - \gamma(1+\gamma)e^{\gamma}\tau_1 - a^2 f(\theta_1(x_0)\tau_1) \\
&\geq \epsilon^q k_1 \tau_1' - \gamma(1+\gamma)e^{\gamma}\frac{f(\theta_1(x_0)\tau_1)}{\theta_1(x_0)} - a^2 f(\theta_1(x_0)\tau_1) \\
&= \epsilon^q k_1 \left\{ \tau_1' - \epsilon^{-q} k_1^{-1} \left[\frac{\gamma(1+\gamma)e^{\gamma}}{\theta_1(x_0)} + a^2 \right] f(\theta_1(x_0)\tau_1) \right\} \\
&= 0.
\end{aligned}
$$

Since $\tau_1(0) = 1, \quad 0 \leq x \leq 1$, and $k_1 = 1 + \max_{x\in\bar{D}}(u_0(x))$,

$$
\begin{aligned}
\theta_1(x) &= (1-x)^\gamma e^{\gamma x} + k_1 \\
&= (1-x)^\gamma e^{\gamma x} + 1 + \max_{x\in\bar{D}}(u_0(x)) > 0.
\end{aligned}
$$

So, we have $\mu_1(x,0) = \theta_1(x)\tau_1(0) > u_0(x)$. Also, $\mu_{1_x}(1,t) = \theta_1'(1)\tau_1(t) = 0$, since $\theta_1'(1) = -\gamma(1-1)^{\gamma-1}e^{\gamma} = 0$. Then $\mu_1(0,t) = \theta_1(0)\tau_1(t) = ((1-0)^\gamma e^{\gamma 0} + k_1)\tau_1(t) > 0$. Since $L\mu_1 - a^2 f(\mu_1(x_0,t)) \geq 0$ and $Lu = a^2 f(u(x_0,t))$, using the Mean Value Theorem (Appendix A), we attain

$$
\begin{aligned}
L(\mu_1 - u) &= L\mu_1 - Lu \\
&\geq a^2 f(\mu_1(x_0,t)) - a^2 f(u(x_0,t)) \\
&= a^2 f'(\zeta(x_0,t))(\mu_1(x_0,t) - u(x_0,t)),
\end{aligned}
$$

for some $\zeta(x_0,t)$ between $\mu_1(x_0,t)$ and $u(x_0,t)$. By Lemma 3.1,

$$\mu_1(x,t)(\in C^{2,1}(\bar{\omega})) \text{ is an upper solution.}$$

□

Let $\rho(x)$ in $C^3[0,1]$ be an increasing function such that $\rho(x)$ is 0 for $x \leq 0$ and 1 for $x \geq 1$. Also, let δ be some positive constant with $\delta < x_0/2$, $\quad D_\delta = (\delta, 1)$, $\quad \omega_\delta = D_\delta \times (0, t_0)$, $\quad \bar{D}_\delta$ and $\bar{\omega_\delta}$ be the closures of D_δ and ω_δ respectively,

$$\rho_\delta = \begin{cases} 0 & \text{for } x \le \delta, \\ \rho(\frac{x}{\delta} - 1) & \text{for } \delta < x < 2\delta, \\ 1 & \text{for } x \ge 2\delta, \end{cases}$$

$$u_{0_\delta}(x) = \rho_\delta(x) u_0(x).$$

From

$$\frac{\partial u_{0_\delta}(x)}{\partial \delta} = \begin{cases} 0 & \text{for } x \le \delta, \\ -\dfrac{x}{\delta^2}\rho'\left(\dfrac{x}{\delta} - 1\right) u_0(x) & \text{for } \delta < x < 2\delta, \\ 0 & \text{for } x \ge 2\delta, \end{cases}$$

we have $\partial u_{0_\delta}(x)/\partial\delta \le 0$, and $u_{0_\delta}(x) \le u_0(x)$.

Let us consider the following problem,

$$\begin{cases} Lu_\delta = a^2 f(u_\delta(x_0, t)) & \text{in} \quad \omega_\delta, \\ u_\delta(x, 0) = u_{0_\delta}(x) (\ge 0) & \text{on} \quad \bar{D}_\delta, \\ u_\delta(\delta, t) = 0 = u_{\delta_x}(1, t) & \text{for} \quad 0 < t < t_0. \end{cases} \tag{3.7}$$

Existence of a classical solution for the problem (3.7) with $u_{\delta_x}(1,t) = 0$ replaced by $u_\delta(1,t) = 0$ has been established by Chan and Yang [CY00]. By using Theorem A.4.1 (instead of Theorem 4.2.2) of Ladde, Lakshmikantham and Vatsala [LLV85], and Theorem 5.3 (instead of Theorem 5.2) of Ladyženskaja, Solonnikov and Ural'ceva [LSU67], a proof similar to that of Theorem 3 of Chan and Yang [CY00] gives the following result.

Lemma 3.3. *The problem (3.7) has a unique nonnegative solution* $u_\delta \in C^{2+\alpha,1+\alpha/2}(\bar{\omega_\delta})$ *such that* $u_\delta(x,t) \le \mu_1(x,t)$.

It is shown in Chan and Liu [CL98] that there exists solution $u_\delta \in C^{2+\alpha,1+\alpha/2}(\bar{\omega_\delta})$ of (3.7).

Let $\lim_{\delta\to 0} u_\delta(x,t) = u(x,t)$. By using the singular index 3 (cf. Ladyženskaja, Solonnikov and Ural'ceva [LSU67]), a proof similar to that of Lemma 2 of Chan and Liu [CL98] gives the following result.

Theorem 3.4. *The problem (1.3) has a unique solution* $u(x,t) \in C(\bar{\omega}) \cap C^{2,1}((0,1] \times [0,t_0])$.

Since $0 \leq u_\delta \leq \mu_1, \lim_{\delta \to 0} u_\delta$ exists for all $(x,t) \in \bar{\omega}$. It is also shown that $u_{\delta_1}, (u_\delta)_t, (u_\delta)_x$, and $(u_\delta)_{xx}$ are equicontinuous (Appendix A) in $\bar{\omega}$. By the Ascoli-Arzela Theorem (Appendix A), the partial derivatives of u are the limits of the corresponding derivatives of u_δ. Therefore, $u(x,t) = \lim_{\delta \to 0} u_\delta$.

Let T be the supremum over t_0 for which the problem (1.3) has a unique solution $u \in C(\bar{\omega}) \cap C^{2,1}((0,1] \times [0,t_0])$. Then, it has a unique solution $u(x,t) \in C(\bar{D} \times [0,T)) \cap C^{2,1}((0,1] \times [0,T))$. We modify the proof of Theorem 2.5 by Floater [Flo91] to prove the following result.

Theorem 3.5. *If* $T < \infty$, *then* $u(x_0,t)$ *is unbounded in* $(0,T)$.

Proof. Let us suppose that $u(x_0,t)$ is bounded above by some positive constant M in Ω. We would like to show that u can be continued into a time interval $[0, T+\tilde{t}_0]$ for some positive $\tilde{t}_0$. To do so, let

$$K = \max\left\{ a^2 f(M), \quad \max_{x \in \bar{D}} \frac{2u_0(x)}{x(2k_3+1-x)} \right\},$$

$$\begin{aligned} W(x) &= \frac{Kx(2k_3+1-x)}{2} \\ &= Kxk_3 + \tfrac{1}{2}Kx - \tfrac{1}{2}Kx^2, \end{aligned}$$

$$\begin{aligned} W'(x) &= Kk_3 + \tfrac{1}{2}K - Kx, \\ W''(x) &= -K, \end{aligned}$$

where $k_3 \geq 1/2$. Since $LW = x^q W_t - W_{xx} = x^q \cdot 0 - (-K) = K$, we have

$$\begin{aligned} L(W-u) &= LW - Lu \\ &= K - a^2 f(u(x_0,t)) \\ &\geq a^2 f(M) - a^2 f(u(x_0,t)) \\ &= a^2 \Big(f(M) - f(u(x_0,t)) \Big) \geq 0 \quad \text{in} \quad \Omega. \end{aligned}$$

Also, we have

$$W(x) \geq \frac{\left(\max_{x\in\bar{D}} \frac{2u_0(x)}{x(2k_3+1-x)}\right)x(2k_3+1-x)}{2} \geq u_0(x).$$

Also,

$$W(0) = \frac{K\cdot 0(2k_3+1-0)}{2} = 0 = u(0,t),$$

and since $k_3 \geq \frac{1}{2}$,

$$W'(1) = Kk_3 + \tfrac{1}{2}K - K = K(k_3 - 0.5) \geq 0, \text{ and } u_x(1,t) = 0,$$

we have $W'(1) = K(k_3 - 0.5) \geq u_x(1,t)$ for $t > 0$.

By Lemma 3.1, $W(x)$ is an upper solution of $u(x,t)$ for $0 \leq t \leq T$.

Taking $W(x)$ as the initial function at $t = T$, we can construct, as in Lemma 3.2, an upper solution $\tilde{\mu}_1(x,t)$ of $u(x,t)$ on $\bar{D}\times[T, T+\tilde{t}_0]$ for some positive $\tilde{t}_0$. Thus, u can be continued into a time interval $[0, T+\tilde{t}_0]$. This contradicts the definition of T. Hence, the theorem is proved. □

Theorem 3.6. *If $u_0(x)$ is sufficiently large in the neighborhood of x_0, then u blows up in a finite time.*

Proof. Let

$$\begin{aligned}
\theta_2(x) &= (x - x_0 + \epsilon)^2(x - x_0 - \epsilon)^2, \\
&= (x^2 - 2x_0x + x_0^2 - \epsilon^2)(x^2 - 2x_0x + x_0^2 - \epsilon^2) \\
&= x^4 + x^3(-2x_0 - 2x_0) + x^2(x_0^2 - \epsilon^2 + 4x_0^2 + x_0^2 - \epsilon^2) \\
&+ x(-2x_0^3 + 2x\epsilon^2x_0 - 2x_0^3 + 2\epsilon^2x_0) + (x_0^4 - \epsilon^2x_0^2 - \epsilon^2x_0^2 + \epsilon^4) \\
&= x^4 + x^3(-4x_0) + x^2(6x_0^2 - 2\epsilon^2) + x(-4x_0^3 + 4\epsilon^2x_0) + (x_0^4 - 2\epsilon^2x_0^2 + \epsilon^4).
\end{aligned}$$

Also,

$$\theta_2'(x) = 4x^3 + 3x^2(-4x_0) + 2x(6x_0^2 - 2\epsilon^2) + (-4x_0^3 + 4\epsilon^2x_0),$$

and

$$\begin{aligned}\theta_2''(x) &= 12x^2 - 24x_0x + 12x_0^2 - 4\epsilon^2 \\ &= 4(3x^2 - 6x_0x + 3x_0^2 - \epsilon^2).\end{aligned}$$

Also, we have

$$\tau_2' + \frac{4}{(x_0+\epsilon)^q\epsilon^2}\tau_2 = \frac{a^2\epsilon^{4r}}{(x_0+\epsilon)^q}\tau_2^{1+r}, \tau_2(0) \equiv \tau_0 > \max\left\{\frac{1}{\epsilon^4}, \left(\frac{4}{a^2\epsilon^{4r+2}}\right)^{\frac{1}{r}}\right\}, \tag{3.8}$$

where ϵ is sufficiently small such that $0 < x_0 - \epsilon < x_0 + \epsilon < 1$, and r is given in (1.4).

We note that

$$\theta_2''(x) = 4(3x^2 - 6x_0x + 3x_0^2 - \epsilon^2)$$

is a quadratic function with vertex at $x = x_0$, $a = 12, b = -24x_0$, and $c = (3x_0^2 - \epsilon^2)4$. Now,

$$\begin{gathered}\theta_2'' = 0 \text{ at } x = x_0 - \frac{\sqrt{3}}{3}\epsilon \text{ and } x = x_0 + \frac{\sqrt{3}}{3}\epsilon, \\ \theta_2'' < 0 \text{ for } x \in \left(x_0 - \frac{\sqrt{3}}{3}\epsilon, x_0 + \frac{\sqrt{3}}{3}\epsilon\right), \\ \theta_2'' > 0 \text{ for } x \in \left(x_0 - \epsilon, x_0 - \frac{\sqrt{3}}{3}\right), \quad x \in \left(x_0 + \frac{\sqrt{3}}{3}, x_0 + \epsilon\right), \\ \theta_2(x_0) = (x_0 - x_0 + \epsilon)^2(x_0 - x_0 - \epsilon)^2 = \epsilon^4\end{gathered}$$

and $\theta_2(x)$ attains its maximum ϵ^4 at $x = x_0$.

Let $\mu_2(x,t) = \theta_2(x)\tau_2(t)$. Then for $x \in (x_0 - \epsilon, x_0 + \epsilon)$, since $f(u) \geq u^{1+r}, u \geq 1$, and $f(\epsilon^4\tau_2) \geq (\epsilon^4\tau_2)^{1+r}$ provided $\epsilon^4\tau_2 \geq 1$, and $\epsilon^4\tau_2 \geq \epsilon^4(1/\epsilon^4) = 1$ from (3.8), since $x_0 + \epsilon$ is maximum, $\theta_2(x_0) = \epsilon^4$, and $\tau_2' + \frac{4}{(x_0+\epsilon)^q\epsilon^2}\tau_2 = \frac{a^2\epsilon^{4r}}{(x_0+\epsilon)^q}\tau_2^{1+r}$,

$$\begin{aligned}&L\mu_2 - a^2 f(\mu_2(x_0,t)) \\ &= L(\theta_2(x)\tau_2(t)) - a^2 f(\mu_2(x_0,t)) \\ &= x^q(x - x_0 + \epsilon)^2(x - x_0 - \epsilon)^2\tau_2' + (-12x^2 + 24x_0x - 12x_0^2 + 4\epsilon^2)\tau_2 - a^2 f(\theta_2(x_0)\tau_2) \\ &\leq x^q(x - x_0 + \epsilon)^2(x - x_0 - \epsilon)^2\tau_2' + 4\epsilon^2\tau_2 - a^2 f(\theta_2(x_0)\tau_2) \\ &\leq (x_0 + \epsilon)^q\epsilon^4\tau_2' + 4\epsilon^2\tau_2 - a^2 f(\epsilon^4\tau_2) \\ &\leq (x_0 + \epsilon)^q\epsilon^4\tau_2' + 4\epsilon^2\tau_2 - a^2\epsilon^{4(1+r)}\tau_2^{1+r} \\ &= (x_0 + \epsilon)^q\epsilon^4\left[\tau_2' + \frac{4}{(x_0+\epsilon)^q\epsilon^2}\tau_2 - \frac{a^2\epsilon^{4r}}{(x_0+\epsilon)^q}\tau_2^{1+r}\right] \\ &= 0.\end{aligned}$$

For $x \in [x_0 - \epsilon, x_0 + \epsilon], \mu_2(x, 0) \geq 0$. Also, $\mu_2(x_0 - \epsilon, t) = 0 = \mu_2(x_0 + \epsilon, t)$. In (3.8), let $z = \tau_2^{-r}$. We obtain a linear equation,

$$z' - \frac{4r}{(x_0+\epsilon)^q \epsilon^2} z + \frac{a^2 r \epsilon^{4r}}{(x_0+\epsilon)^q} = 0.$$

For solving this, we start from (3.8)

$$\tau_2' + \frac{4}{(x_0+\epsilon)^q \epsilon^2} \tau_2 = \frac{a^2 \epsilon^{4r}}{(x_0+\epsilon)^q} \tau_2^{1+r}.$$

Let $y = \tau_2, a = \frac{4}{(x_0+\epsilon)^q \epsilon^2}$, and $b = \frac{a^2 \epsilon^{4r}}{(x_0+\epsilon)^q}$, then $y' + ay = by^{1+r}$. We have $\frac{dy}{dx} + ay = by^{1+r}$.

Now, $P(x) = a, Q(x) = b, n = 1 + r$, and $v = y^{1-1-r} = y^{-r}$, then since $y = v^{-\frac{1}{r}}$, we have $by^{1+r} = b(v^{-\frac{1}{r}})^{1+r} = bv^{-\frac{1}{r}-1}$, and $dy/dx = (dy/dv)(dv/dx) = -1/rv^{-\frac{1}{r}-1} dv/dx$,

$$-\frac{1}{r} v^{-\frac{1}{r}-1} \frac{dv}{dx} + av^{-\frac{1}{r}} = bv^{-\frac{1}{r}-1},$$

if we divide by $-1/r(v)^{-1/r-1}$ to both sides, then we have

$$\frac{dv}{dx} - arv = -br.$$

Now, we have

$$\begin{aligned}
&\rho(x) = e^{\int -ardx} = e^{-arx}, \\
&D_x(e^{-arx} v) = -br(e^{-arx}), \\
&e^{-arx} v = \int -br(e^{-arx}) dx = \frac{b}{a}(e^{-arx}) + c, \\
&e^{-arx} y^{-r} = \frac{b}{a}(e^{-arx}) + c, \\
&y^{-r} = \frac{b}{a} + \frac{c}{e^{-arx}}.
\end{aligned}$$

Since $\tau_2 = y, a = \frac{4}{(x_0+\epsilon)^q \epsilon^2}, b = \frac{a^2 \epsilon^{4r}}{(x_0+\epsilon)^q}$, and $x = t$.

$$\begin{aligned}
\tau_2^{-r} &= \frac{\frac{a^2 \epsilon^{4r}}{(x_0+\epsilon)^q}}{\frac{4}{(x_0+\epsilon)^q \epsilon^2}} + \frac{c}{e^{-\frac{4rt}{(x_0+\epsilon)^q \epsilon^2}}} \\
&= \frac{a^2 \epsilon^{4r+2}}{4} + c e^{\frac{4rt}{(x_0+\epsilon)^q \epsilon^2}},
\end{aligned}$$

$$\tau_2 = \left[\frac{a^2\epsilon^{4r+2}}{4} + ce^{\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}}\right]^{-\frac{1}{r}},$$

since $e^{\left(\frac{4\cdot r\cdot 0}{(x_0+\epsilon)^q\epsilon^2}\right)} = e^0 = 1$, now we have

$$\begin{aligned}
\tau_0 = \tau_2(0) &= \left[\frac{a^2\epsilon^{4r+2}}{4} + c\right]^{-\frac{1}{r}},\\
\tau_2^{-r} &= \frac{a^2\epsilon^{4r+2}}{4} + c,\\
\tau_2^{-r} - \frac{a^2\epsilon^{4r+2}}{4} &= c,\\
\tau_2 &= \left[\frac{a^2\epsilon^{4r+2}}{4} + \left(\frac{1}{\tau_0^r} - \frac{a^2\epsilon^{4r+2}}{4}\right)e^{\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}}\right]^{-\frac{1}{r}}.
\end{aligned}$$

Therefore, $\tau_2 = \left\{\left[\frac{a^2\epsilon^{4r+2}}{4}e^{-\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}} + \frac{1}{\tau_0^r} - \frac{a^2\epsilon^{4r+2}}{4}\right]e^{\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}}\right\}^{-\frac{1}{r}}$.

To find the blow-up time,

$$\begin{aligned}
&\frac{a^2\epsilon^{4r+2}}{4}e^{-\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}} + \frac{1}{\tau_0^r} - \frac{a^2\epsilon^{4r+2}}{4} = 0,\\
&\frac{a^2\epsilon^{4r+2}}{4}e^{-\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}} = -\frac{1}{\tau_0^r} + \frac{a^2\epsilon^{4r+2}}{4},\\
&\frac{a^2\epsilon^{4r+2}}{4}e^{-\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}} = \frac{-4 + a^2\tau_0^r\epsilon^{4r+2}}{4\tau_0^r},\\
&e^{-\frac{4rt}{(x_0+\epsilon)^q\epsilon^2}} = \frac{\frac{-4 + a^2\tau_0^r\epsilon^{4r+2}}{4\tau_0^r}}{\frac{a^2\epsilon^{4r+2}}{4}} = \frac{-4 + a^2\tau_0^r\epsilon^{4r+2}}{\tau_0^r a^2\epsilon^{4r+2}} = 1 - \frac{4}{a^2\tau_0^r\epsilon^{4r+2}},\\
&\frac{-4rt}{(x_0+\epsilon)^q\epsilon^2} = \ln\left(1 - \frac{4}{a^2\tau_0^r\epsilon^{4r+2}}\right).
\end{aligned}$$

Therefore, $t = \frac{-(x_0+\epsilon)^q\epsilon^2}{4r}\ln\left(1 - \frac{4}{a^2\tau_0^r\epsilon^{4r+2}}\right)$.

Since $\tau_2(0) \equiv \tau_0 > \max\left\{\frac{1}{\epsilon^4}, \left(\frac{4}{a^2\epsilon^{4r+2}}\right)^{\frac{1}{r}}\right\}$, then $\frac{4}{a^2\tau_0^r\epsilon^{4r+2}} < 1$.

Therefore, $\tau_2(t)$ blows up at the time $t_{b_1} = -\frac{(x_0+\epsilon)^q\epsilon^2}{4r}\ln\left(1 - \frac{4}{a^2\tau_0^r\epsilon^{2+4r}}\right) > 0$.

Since

$$Lu = a^2 f(u(x_0,t)) = a^2 f'(\xi)u(x_0,t),$$

where ξ lies between 0 and $u(x_0,t)$, it follows from Lemma 3.1 that $u(x,t) \geq 0$. Therefore, if we choose $u_0(x)$ such that $u_0(x) \geq \mu_2(x,0)$ on $[x_0-\epsilon, x_0+\epsilon]$, then by Lemma 1 of

Chan and Yang [CY00], $u(x,t) \geq \mu_2(x,t)$. Hence, the solution $u(x,t)$ of the problem (1.3) blows up no later than t_{b_1} whenever $u_0(x) \geq \mu_2(x,0)$.

□

3.3 Complete Blow-Up

Green's function $G(x,t;\xi,\tau)$ corresponding to the problem (1.3) is determined by the following system: for x and ξ in D, and t and τ in $(-\infty,\infty)$,

$$\begin{aligned} &LG = \delta(x-\xi)\delta(t-\tau), \\ &G(x,t;\xi,\tau) = 0 \quad \text{for} \quad t < \tau, \\ &G(0,t;\xi,\tau) = 0 = G_x(1,t;\xi,\tau), \end{aligned}$$

where $\delta(x)$ is the Dirac delta function.

Definition of the *Dirac delta function* δ:
We define the *delta function*, or more accurately the *delta distribution* [McO97], in $\mathbf{R}^n$ to be object $\delta(x)$ so that formally

$$\int_{\mathbf{R}^n} \delta(x)v(x)dx = v(0)$$

for every test function $v \in C_0^\infty(\mathbf{R}^n)$. $C_0^\infty(\mathbf{R}^n)$ denotes the space of continuous functions with continuous derivatives on $\mathbf{R}^n$ whose support is a compact subset of $\mathbf{R}^n$. The support of a continuous function $f(x)$ defined on $\mathbf{R}^n$ is the closure of the set of points where $f(x)$ is nonzero [McO97]. In the one-dimensional case we find $H'(x) = \delta(x)$, where the Heaviside function of a single real variable is:

$$H(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0. \end{cases}$$

Notice that we can take any number of distributional derivatives of $\delta(x)$. Also, we can translate the singularity in $\delta(x)$ to any point $\mu \in \mathbf{R}^n$ by letting $\delta_\mu(x) = \delta(x-\mu)$ so that a change of variables $y = x - \mu$ yields

$$\int_{\mathbf{R}^n} \delta_\mu v(x)dx = \int_{\mathbf{R}^n} \delta(x-\mu)v(x)dx = \int_{\mathbf{R}^n} \delta(y)v(y+\mu)dx = v(\mu).$$

Lemma 3.7. *If h(x,t) is nontrivial (not indentically zero) such that* $0 \le h(x,t) \le 1$ *and* $h(x,t) \in C^{\infty}(\boldsymbol{R}^2)$, *then for any finite* θ, *the degenerate linear parabolic problem,*

$$\begin{cases} Lv = h(x,t) \quad in \quad D \times (0,\theta], \\ v(x,0) = 0 \quad on \quad \bar{D}, \\ v(0,t) = 0 = v_x(1,t) \quad for \quad 0 < t \le \theta, \end{cases} \tag{3.9}$$

has a unique solution $v(x,t) \in C(\bar{D} \times [0,\theta]) \cap C^{2,1}((0,1] \times [0,\theta])$.

Proof. Let δ be some positive constant such that $\delta < x_0$. We consider the problem,

$$\begin{cases} Lv_\delta = h(x,t) \quad \text{in} \quad D_\delta \times (0,\theta], \\ v_\delta(x,0) = 0 \quad \text{on} \quad \bar{D}_\delta, \\ v_\delta(\delta,t) = 0 = v_{\delta_x}(1,t) \quad \text{for} \quad 0 < t \le \theta. \end{cases} \tag{3.10}$$

We would like to construct an upper solution $\mu_3(x,t) \in C^{2,1}(\bar{D} \times [0,\theta])$ in the form $\theta_3(x)\tau_3(t)$ for v and all v_δ given by (3.9) and (3.10) respectively. Let $\theta_3(x) = xe^{1-x}$, $\theta_3'(x) = e^{1-x} - xe^{1-x}$, $\theta_3''(x) = e^{1-x}(x-2)$, and ϵ be a fixed positive number less than x_0, and

$$\tau_3' = \epsilon^{-q}(1+\epsilon^{-1})\tau_3, \tau_3(0) = 1.$$

It follows that

$$\tau_3(t) = e^{\epsilon^{-q}(1+\epsilon^{-1})t} \ge 0,$$

which is increasing and bounded for any $t \le \theta$.

Let $\mu_3(x,t) = \theta_3(x)\tau_3(t)$. Then,

$$\begin{aligned} \mu_3(x,t) &= \theta_3(x)\tau_3(t) \\ &= (xe^{1-x})(e^{\epsilon^{-q(1+\epsilon^{-1})t}}) \in C^{2,1}(\bar{D} \times [0,\theta]). \end{aligned}$$

For any $x \in [0,\epsilon]$, and any $t \le \theta$,

$$\begin{aligned} L\mu_3 - h(x,t) &= L(\theta_3(x)\tau_3(t)) - h(x,t) \\ &= x^q(xe^{1-x})\epsilon^{-q}(1+\epsilon^{-1})\tau_3 - e^{1-x}(x-2)\tau_3 - h(x,t) \\ &\ge -e^{1-x}(x-2)\tau_3 - 1 \\ &\ge e^{1-x}\tau_3\left(-\epsilon + 2 - \frac{1}{e^{1-x}\tau_3}\right) \\ &\ge e^{1-x}\tau_3(-\epsilon + 2 - 1) \\ &= (1-\epsilon)e^{1-x}\tau_3 > 0. \end{aligned}$$

Since $\tau_3(t)$ is an increasing function, we have for any $x \in (\epsilon, 1]$ and any $t \leq \theta$,

$$\begin{aligned} L\mu_3 - h(x,t) &= x^q(xe^{1-x})\epsilon^{-q}(1+\epsilon^{-1})\tau_3 - e^{1-x}(x-2)\tau_3 - h(x,t) \\ &\geq \epsilon^q x e^{1-x}\tau_3' - xe^{1-x}\tau_3 - 1 \\ &= \epsilon^q x e^{1-x}\left(\tau_3' - \epsilon^{-q}\tau_3 - \frac{\epsilon^{-q}\tau_3}{e^{1-x}x\tau_3}\right) \\ &\geq \epsilon^q x e^{1-x}\left(\tau_3' - \epsilon^{-q}\tau_3 - \frac{\epsilon^{-q}\tau_3}{e^{1-x}\epsilon\tau_3}\right) \\ &\geq \epsilon^q x e^{1-x}(\tau_3' - \epsilon^{-q}\tau_3 - \epsilon^{-q-1}\tau_3) \\ &= 0. \end{aligned}$$

Since $\mu_3(x,0) = \theta_3(x)\tau_3(0) = xe^{1-x}\cdot 1 = xe^{1-x} \geq 0$, $\mu_3(0,t) = \theta_3(0)\tau_3(t) = 0$, and $\mu_{3_x}(1,t) = \theta_3'(1)\tau_3(t) = (e^{1-1} - e^{1-1})\tau_3(t) = 0$, it follows from the strong maximum principle [Fri64] and the parabolic version of Hopf's lemma [Fri64] that $\mu_3(x,t)$ is an upper solution for all v_δ and v.

We note that $x^{-q} \in C^{\alpha,\alpha/2}(\bar{D}_\delta \times [0,\theta])$, $|x^{-q}h| \leq \delta^{-q}$ for $(x,t,v_\delta) \in \bar{D}_\delta \times [0,\theta] \times R$, and $v_{0_\delta}(x) = 0 \in C^{2+\alpha}(\bar{D}_\delta)$.

Definition of *Hölder Continuity*:

If $0 \leq \alpha \leq 1$ and u is defined and continuous in a neighborhood U of x_0, then we can say that u is *Hölder continuous* at x_0 with exponent α if

$$[u]_{\alpha;x_0} \equiv \sup_{x\in U}\frac{|u(x)-u(x_0)|}{|x-x_0|^\alpha} < \infty.$$

Here $x_0 \in \bar{D}_\delta = [\delta, 1]$.

Let us prove that $x^{-q} \in C^{\alpha,\alpha/2}(\bar{D}_\delta \times [0,\theta])$. By the mean value theorem, $f(u) - f(v) = f'(\xi)(u-v)$. Let $f(x) = x^{-q}$, and $f(x_0) = x_0^{-q}$,

$$\begin{aligned} \left|x^{-q} - x_0^{-q}\right| &= |f'(\xi)(x-x_0)| \\ &= \left|-q\xi^{-q-1}(x-x_0)\right| \\ &= \frac{q}{\xi^{q+1}}(x-x_0) < \infty. \end{aligned}$$

So

$$\begin{aligned} [u]_{\alpha;x_0} &\equiv \sup_{x\in U}\frac{|u(x)-u(x_0)|}{|x-x_0|^\alpha} \\ &\equiv \sup_{x\in U}\frac{\left|x^{-q}-x_0^{-q}\right|}{|x-x_0|^\alpha} < \infty. \end{aligned}$$

By Theorem A.4.1 of Ladde, Lakshmikantham and Vatsala [LLV85] (Appendix A), the problem (3.10) has a unique solution $v_\delta \in C^{2+\alpha,1+\alpha/2}(\bar{D}_\delta \times [0,\theta])$.

By the strong maximum principle (Appendix A), $v_\delta \geq 0$ on $\bar{D}_\delta \times [0,\theta]$. Let (A), (B), (C) hold. $Lu \leq 0$ means $u \geq 0$, so $Lv_\delta \leq 0$ means $v_\delta \geq 0$, on $\bar{D}_\delta \times [0,\theta]$.
For $\delta_1 \leq \delta_2$,

$$L(v_{\delta_1} - v_{\delta_2}) = 0 \text{ for } x \in (\delta_2, 1)$$
$$v_{\delta_1}(x,0) = v_{\delta_2}(x,0) \text{ for } x \in [\delta_2, 1]$$
$$v_{\delta_1}(\delta_2,t) - v_{\delta_2}(\delta_2,t) \geq 0, \quad v_{\delta_{1_x}}(1,t) - v_{\delta_{2_x}}(1,t) = 0, \quad 0 < t \leq \theta.$$

By Lemma 3.1, $v_{\delta_1} \geq v_{\delta_2}$ on $\bar{D}_{\delta_2} \times [0,\theta]$, Thus, $\lim_{\delta \to 0} v_\delta$ exists, since v_δ is bounded by upper solution μ_3 and is monotone because $v_{\delta_1} \geq v_{\delta_2}$.

Let $v(x,t) = \lim_{\delta \to 0} v_\delta(x,t)$. A proof similar to that of Theorem 3.5 shows that $v(x,t)$ is a solution of the problem (3.9), and $v(x,t) \in C(\bar{D} \times [0,\theta]) \cap C^{2,1}((0,1] \times [0,\theta])$.

Let's prove that $v(x,t)$ is unique. Let $y_1 = v_1 - v_2$, and $y_2 = v_2 - v_1$. We have

$$\begin{cases} Lv_1 = x^q v_{1_t} - v_{1_{xx}} = h(x,t) \quad \text{in} \quad D \times (0,\theta], \\ v_1(x,0) = 0 \quad \text{on} \quad \bar{D}. \\ v_1(0,t) = 0 = v_{1_x}(1,t) \quad \text{for} \quad 0 < t \leq \theta. \end{cases} \tag{3.11}$$

$$\begin{cases} Lv_2 = x^q v_{2_t} - v_{2_{xx}} = h(x,t) \quad \text{in} \quad D \times (0,\theta], \\ v_2(x,0) = 0 \quad \text{on} \quad \bar{D}. \\ v_2(0,t) = 0 = v_{2_x}(1,t) \quad \text{for} \quad 0 < t \leq \theta. \end{cases} \tag{3.12}$$

Subtracting equations (3.12) from (3.11), we get

$$\begin{cases} Ly_1 = L(v_1 - v_2) = 0 \\ v_1(x,0) - v_2(x,0) = 0, \\ v_1(0,t) - v_2(0,t) = 0, \\ v_{1_x}(1,t) - v_{2_x}(1,t) = 0. \end{cases}$$

Subtracting equations (3.11) from (3.12), we get

$$\begin{cases} Ly_2 = L(v_2 - v_1) = 0 \\ v_2(x,0) - v_1(x,0) = 0, \\ v_2(0,t) - v_1(0,t) = 0, \\ v_{2_x}(1,t) - v_{1_x}(1,t) = 0. \end{cases}$$

By Lemma 3.1, $y_1 \geq 0$, then $v_1 - v_2 \geq 0$, so $v_1 \geq v_2$, and $y_2 \geq 0$, then $v_2 - v_1 \geq 0$, so $v_2 \geq v_1$, hence $v_1 \geq v_2$ and $v_2 \geq v_1$, thus $v_1 = v_2$. Therefore, $v(x,t)$ is a unique solution. □

Lemma 3.8. *Given any $x \in (0,1]$ and any finite time θ, there exist positive constants k_4 (depending on x and θ) and k_5 (depending on θ) such that*

$$k_4 < \int_0^1 G(x,t;\xi,\tau)d\xi \quad for \quad 0 \leq t \leq \theta,$$
$$\int_0^1 G(x_0,t;\xi,\tau)d\xi < k_5 \quad for \quad 0 \leq t \leq \theta.$$

Proof. By Lemma 3.7, the problem,

$$Lv = 1 \text{ in } D \times (0,\theta], v(x,0) = 0 \text{ on } \bar{D}, v(0,t) = 0 = v_x(1,t) \text{ for } 0 < t \leq \theta,$$

has a unique solution $v \in C(\bar{D} \times [0,\theta]) \cap C^{2,1}((0,1] \times [0,\theta])$. The adjoint operator L^* of L is given by

$$L^*u = -x^q u_t - u_{xx}.$$

Definition: The *operator* L^* is called the *adjoint* of L, and is an m-th order linear differential operator with continuous coefficients.

Let $Lu = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u$, where $a_\alpha \in C^{|\alpha|}(\Omega)$, $u \in C^m(\Omega)$, and $v \in C_0^m(\Omega)$. Then $\int_\Omega (D^\alpha u) v dx = (-1)^m \int_\Omega u D^\alpha v dx$, where $m = |\alpha|$,

$$L^*v = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha(x) v).$$

Therefore, $\int_\Omega (Lu) v dx = \int_\Omega u(L^*v) dx$ [McO97].

Using Green's second identity, we obtain

$$v(x,t) = \int_0^t \int_0^1 G(x,\xi,t-\tau)d\xi d\tau = \int_0^t \int_0^1 G(x,\xi,t)d\xi d\tau.$$

This gives

$$v_t = \int_0^1 G(x,\xi,t)d\xi,$$

corresponding to the problem

$$\begin{cases} Lv_t = 0 \quad \text{in} \quad \Omega, \\ v_t(x,0) = x^{-q} \quad \text{for} \quad 0 < x \leq 1, \\ v_t(0,t) = 0 = v_{t_x}(1,t) \quad \text{for} \quad 0 < t < \infty. \end{cases}$$

By using the strong maximum principle [Fri64] and the parabolic version of Hopf's lemma [Fri64] (Appendix A), $v_t > 0$ in $(0,1] \times [0,\theta]$. Thus for any finite time θ, there exists a positive constant k_4 (depending on x and θ) such that

$$\int_0^1 G(x,\xi,t)d\xi > k_4 \text{ for } 0 \leq t \leq \theta.$$

Since $v(x,t) \in C(\bar{\Omega}) \cap C^{2,1}((0,1] \times [0,\theta])$, there exists a positive constant k_5 (depending on θ) such that

$$\int_0^1 G(x_0,t;\xi)d\xi < k_5 \text{ for } 0 \leq t \leq \theta.$$

□

Our next result gives the complete blow-up of the solution u.

Theorem 3.9. *If the solution of the problem (1.3) blows up in a finite time t_b, then the blow-up set is $\bar{D}$.*

Proof. By Green's second identity,

$$u(x,t) = \int_0^t \int_0^1 a^2 f(u(x_0,\tau))G(x,\xi,t-\tau)d\xi d\tau + \int_0^t \xi^q u_0(\xi)G(x,\xi,t)d\xi,$$

for any $t < t_b$. If the solution of the problem (1.3) blows up in a finite time t_b, then by Theorem 3.5, u blows up at the point x_0. We know that maximum of ξ^q is 1, maximum of $\int_0^1 u_0(\xi)d\xi$ is $\max_{x\in\bar{D}} u_0(x)$, $\int_0^1 G(x_0,t;\xi)d\xi < k_5$, for $0 \leq t \leq \theta$, and $a^2 \int_0^t \int_0^1 G(x_0,\xi,\tau)f(u(x_0,t-\tau))d\xi d\tau \leq k_5 a^2 \int_0^t f(u(x_0,t-\tau))d\tau$. It follows from Lemma 3.8 that for any $t < t_b$,

$$\begin{aligned} u(x_0,t) &= \int_0^1 \xi^q G(x_0,\xi,t)u_0(\xi)d\xi + a^2 \int_0^t \int_0^1 G(x_0,\xi,\tau)f(u(x_0,t-\tau))d\xi d\tau \\ &\leq k_5 \left(\max_{x\in\bar{D}} u_0(x) + a^2 \int_0^t f(u(x_0,t-\tau))d\tau \right). \end{aligned}$$

Since $u(x_0,t) \to \infty$ as $t \to t_b^-$, we have

$$\int_0^{t_b} f(u(x_0, t_b - \tau))d\tau = \infty.$$

On the other hand, for any $(x,t) \in (0,1] \times [0,t_b)$, since $\int_0^1 G(x,\xi,t)d\xi > k_4$ for $0 \le t \le \theta$,
we have

$$\begin{aligned} u(x,t) &= \int_0^1 \xi^q G(x,\xi,t)u_0(\xi)d\xi + a^2 \int_0^t \int_0^1 G(x,\xi,\tau)f(u(x_0,t-\tau))d\xi d\tau \\ &> a^2 \int_0^t \int_0^1 G(x,\xi,\tau)f(u(x_0,t-\tau))d\xi d\tau \ge a^2 k_4 \int_0^t f(u(x_0,t-\tau))d\tau, \end{aligned}$$

which tends to infinity as t approaches t_b^-. For $x = 0$, we can always find a sequence $\{(x_n,t_n)\}$ such that $(x_n,t_n) \to (0,t_b)$ and $\lim_{n\to\infty} u(x_n,t_n) \to \infty$. Thus, the blow-up set is $\bar{D}$.

□

Appendix A

1. **The Maximum Principle** [Fri64]

   Consider the operator

$$Nu \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t}$$

   in an $(n+1)$-dimensional domain Ω with the following assumptions:

   (A) N is parabolic in Ω, i.e., for every $(x,t) \in \Omega$ and for any real vector $\xi \neq 0, \sum a_{ij}(x,t)\xi_i\xi_j > 0$;

   (B) the coefficients of N are continuous functions in Ω;

   (C) $c(x,t) \leq 0$ in Ω.

   The functions u are always assumed to have two continuous x-derivatives and one continuous t-derivative in Ω.

   Definition A.1. *Notation: For any point $P^0 = (x^0, t^0)$ in D, we denote by $S(P^0)$ the set of all points Q in D which can be connected to P^0 by a simple continuous curve in D along which the t-coordinate is nondecreasing from Q to P^0. By $C(P^0)$, we denote the component (in $t = t^0$) of $D \cap \{t = t^0\}$ which contains P^0. Note that $S(P^0) \supset C(P^0)$.*

   Theorem A.2. *Let (A), (B), (C) hold. If $Lu \geq 0 \quad (Lu \leq 0)$ in D and if u has in D a positive maximum (negative minimum) which is attained at a point $P^0(x^0, t^0)$, then $u(P) = u(P^0)$ for all $P \in S(P^0)$.*

2. **Extensions of the Maximum Principle** [Fri64]

Theorem A.3. *Let (A), (B) hold. If* $u \leq 0 \quad (u \geq 0)$ *in* $S(P^0), Lu \geq 0 \quad (Lu \leq 0)$ *in* $S(P^0)$ *and* $u(P^0) = 0$, *then* $u \equiv 0$ *in* $S(P^0)$.

3. **Hopf's Lemma** [Fri64]

Definition A.4. *Let* $P^0 = (x^0, t^0)$ *be a point on the boundary* $\partial\Omega$ *of a domain* Ω. *If there exists a closed ball* B *with center* $(\bar{x}, \bar{t})$ *such that* $B \subset \bar{\Omega}, B \cap \partial\Omega = \{P^0\}$, *and if* $\bar{x} \neq x^0$, *then we say that* P^0 *has the inside strong sphere property.*

Lemma A.5. *Let* P^0 *have the inside strong sphere property. Assume further that, for some neighborhood* V *of* $P^0, u < M$ *in* $D \cap V$. *Then, for any non-tangential inward direction* τ,

$$\frac{\partial u}{\partial \tau} \equiv lim_{\Delta\tau \to 0} inf\left(\frac{\Delta u}{\Delta \tau}\right) < 0 \quad at\ P^0.$$

By a non-tangential inward direction we mean direction pointing from P^0 *into the interior of the ball* B *whose boundary touches* ∂D *at* P^0.

4. **The Mean Value Theorem** [LHE06]

Theorem A.6. *Let* f *be a continuous function on* $[a, b]$ *that is differentiable on* (a, b). *Then there exists at least one point* $c \in (a, b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

5. **Parabolic Equations** [LLV85]

Let $\mathfrak{L}$ be a differential operator defined by

$$\mathfrak{L} = -N,$$

where N is defined on page 36. Let $f \in C^{\alpha/2,\alpha}[[0,T] \times \bar{\Omega} \times R \times R^m, R]$, that is $f(t, x, u, y)$ is Hölder continuous in t and (x, u, y) with exponent $\alpha/2$ and α, respectively, where $0 < \alpha < 1$.

Consider the linear second order parabolic initial boundary value problem (IBVP for short)

$$\begin{cases} \mathfrak{L}u = F(t,x), \quad (t,x) \in Q_T, \\ (Bu)(t,x) = \phi(t,x), \quad (t,x) \in \Gamma_T, \\ u(0,x) = \phi_0(x), \quad x \in \bar{\Omega}, \end{cases}$$

where B is defined by $Bu = p(t,x)u + q(t,x)du/d\nu$ and $du/d\nu$ stands for the normal derivative of u; Ω is a bounded domain; $Q_T = (0,T) \times \Omega$; $\Gamma_T = (0,T) \times \partial\Omega$; and $D_x u = (\partial u/\partial x_1, ..., \partial u/\partial x_m)$.

Theorem A.7. *Assume that*

(a1) a_{ij}, b_i, c *and* $F \in C^{\alpha/2,\alpha}[\bar{\Omega}_T, R], c(t,x) \leq 0$ *and* $\mathfrak{L}$ *is strictly uniformly parabolic in* Q_T*;*

(a2) $p, q \in C^{(1+\alpha)/2,1+\alpha}[\bar{\Gamma}_T, R], p$ *and* q *are nonnegative functions which do not vanish simultaneously;*

(a3) $\partial\Omega$ *belongs to class* $C^{2+\alpha}$*;*

(a4) $\phi \in C^{(1+\alpha)/2,1+\alpha}[\bar{\Gamma}_T, R]$ *and* $\phi_0 \in C^{2+\alpha}[\bar{\Omega}, R]$*;*

(a5) the IBVP

$$\begin{cases} \mathfrak{L}u = F(t,x), \quad (t,x) \in Q_T, \\ (Bu)(t,x) = \phi(t,x), \quad (t,x) \in \Gamma_T, \\ u(0,x) = \phi_0(x), \quad x \in \bar{\Omega}, \end{cases}$$

satisfies the compatibility condition of order $[(1+\alpha)/2]$.

Then this linear parabolic IBVP has a unique solution u *such that* $u \in C^{1+\alpha/2,2+\alpha}[\bar{Q}_T, R]$..

6. The Ascoli-Arzela Theorem [Col88]

Definition A.8. ***: Equicontinuity***

A set of functions $\{\psi_j(s)\}$ *is said to be equicontinuous on an interval* $[0,l]$ *if for every* $\epsilon > 0$ *there exists a number* $\delta = \delta(\epsilon)$ *independent of* j *such that*

$$|\psi_j(s_1) - \psi_j(s_2)| < \epsilon \quad for \quad s_1, s_2 \in [0,l], \quad |s_1 - s_2| < \delta.$$

Definition A.9. *A set of functions* $\{\psi_j(s)\}$ *defined on* $[0, l]$ *is said to be uniformly bounded if there exists a constant* M *independent of* j *such that*

$$max_{0 \leq s \leq l} |\psi_j(s)| \leq M.$$

Theorem A.10. *Let* $\{\psi_j(s)\}$ *be a set of uniformly bounded and equicontinuous functions defined on an interval* $[0, l]$. *Then there exists a subsequence of* $\{\psi_j(s)\}$ *that is uniformly convergent on* $[0, l]$.

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